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Bearings-Only Tracking: A Hybrid Coordinate System Approach

Walter Grossman
Madison Guidance, Navigation, and Control, Yorktown Heights, New York 10598

In this paper a bearings-only tracking algorithm is described. The algorithm is an extended Kalman filter (EKF) which combines the linear-in-state properties of the Cartesian state variable definition with the linear-in-measurement properties of the modified polar (MP) state variable definition. This hybrid approach employs the Cartesian system for state and state covariance extrapolation and employs the MP system for state and state covariance updating. Accurate state and state covariance extrapolation is achieved without numerical integration. The filter equations of this EKF are, furthermore, derived using a new line-of-sight algebra which yields equations which are nonlinear algebraic rather than transcendental. This new formulation obviates the need for real-time calculation of inverse trigonometric function and allows for the straightforward derivation of the three-dimensional problem. In comparison to previous formulations, this approach allows for easier manipulation of the filter equations and provides for greater insight into bearings-only tracking problem.

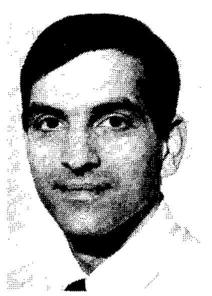
Nomenclature

a = system measurement matrix, columns are unit length and in the direction of the measurement axes
C = MP observation matrix
f_c^{MP} = Cartesian-to-MP state mapping function
f_{MP}^c = MP-to-Cartesian state mapping function
$h(x)$ = generalized state observation function
J_c^{MP} = Jacobian for linear mapping Cartesian state covariance to MP state covariance, $\partial x_{MP}/\partial x_c$
J_{MP}^c = Jacobian for linear mapping MP state covariance to Cartesian state covariance, $\partial x_c/\partial x_{MP}$
K = MP Kalman gain matrix
M = confidence level in percent
P_c = Cartesian state covariance matrix
P_{MP} = MP state covariance matrix
R = LOS vector from own-ship platform to target
$\dot{R}$ = dR/dt, the vector time derivative of the LOS vector
$|R|$ = length of LOS vector, i.e., the range to the target
r = unit vector in the direction of the LOS, $R/|R|$
$\dot{r}$ = dr/dt, the time derivative of the unit LOS vector
T = sample period
t_i = specific time for which the ensemble of range errors from Monte Carlo simulations is sorted
u_{os} = known own-ship acceleration
u_{tg} = unknown target acceleration, acceleration modeled as Gaussian process, i.e., $E[u] = 0$, $E[uu^T] = U$
w = Gaussian measurement noise, $E[w] = 0$, $E[ww^T] = W$
x_c = Cartesian state vector
x_{MP} = MP state vector
y = system measurement, $h(x)$
Γ = Cartesian input distribution matrix
$\delta(\cdot)$ = uncertainty associated with given quantity
$\epsilon_{M\%}$ = minimum range error below which are $M\%$ of range errors from the Monte Carlo simulation ensemble
Φ = Cartesian state transition matrix
$(\hat{\ })$ = signifies a posterior quantity
$(\tilde{\ })$ = signifies a priori quantity

I. Introduction

THE bearings-only tracking problem is of interest for the passive tracking of fixed and moving targets. The problem is often approached by employing some form of an extended Kalman filter (EKF). Since the problem is fundamentally nonlinear, anomalous behavior (e.g., filter lockup or divergence) associated with premature state covariance collapse has been described. The problem is further aggravated by the typically large error in the range estimate used to initialize the filter.

Due to this anomalous behavior, research has centered on the effect of coordinate system choice upon filter stability and performance.[1–3] The Cartesian system provides for linear state extrapolation and nonlinear state updates.[1,2] The polar or modi-



Walter Grossman received his Bachelor's degree in Electrical Engineering from the Massachusetts Institute of Technology, Cambridge, Massachusetts, and a Master's degree and Degree of Engineer, both in Electrical Engineering, from Polytechnic University, Brooklyn, New York. His research interests include application of modern control theory to signal processing, nonlinear estimation, tracking, navigation, and spacecraft attitude determination & control. He is President and Chief Scientist for the consulting firm Madison Guidance, Navigation & Control. He is a member of the AAS, the AIAA and the IEEE.

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fied polar (MP) system[3] provides for nonlinear state extrapolation and linear state updates.

The Cartesian system provides for simplicity in extrapolation at the expense of utilizing a nonlinear measurement equation requiring approximations in the expansion of the observation partial derivatives about the a priori state estimate. The Cartesian EKFs are usually iterated about the update to reduce the deleterious effects of approximation errors.

The MP system provides for a nonlinear but numerically precise, computable, and accurate state extrapolation, a linearized state covariance extrapolation, and a simple linear measurement update. The nonlinear approximations associated with MP system are contained in the extrapolation of the state covariance.

The hybrid system proposed here combines linear extrapolation of the state and state covariance of the Cartesian system with the linear updating of the MP system. This approach provides for accurate extrapolation of the state covariance.

The equations for this hybrid-system, bearings-only filter are developed using an algebraic approached introduced by this author.[4] It was shown[4] that the filter equations are greatly simplified by modifying the measurement equation such that the measured quantity is the direction cosines of the unit line-of-sight (LOS) vector projected onto the known measurement axes rather than the angles between the LOS vector and the axes. The resulting filter equations are nonlinear algebraic rather than transcendental. This formulation, in comparison to the conventional trigonometric formulation, obviates the need to perform inverse trigonometric calculations in real time, provides for straightforward application to the three-dimensional tracking problem and the two-axis measurement problem, provides for easier manipulation of the filter equations, and provides greater insight into the problem.[4] An additional consequence of the algebraic formulation not discussed previously is that the measurement error covariance of the direction cosine is constant over the entire span of measurement angles.

II. Problem Statement

Assume that the own-ship platform has a sensor capable of measuring the angle between the LOS vector and a known axis or axes. Assume furthermore that the navigation information of the own ship is perfect and that the dynamic state of the own ship (the position, velocity, acceleration, attitude, and angular velocity) relative to some inertial coordinate system is known. Assume that a target is detected and that its dynamic state (position and velocity) are unknown. The problem is to determine dynamic state of the target from LOS angle measurements. Figure 1 illustrates the problem.

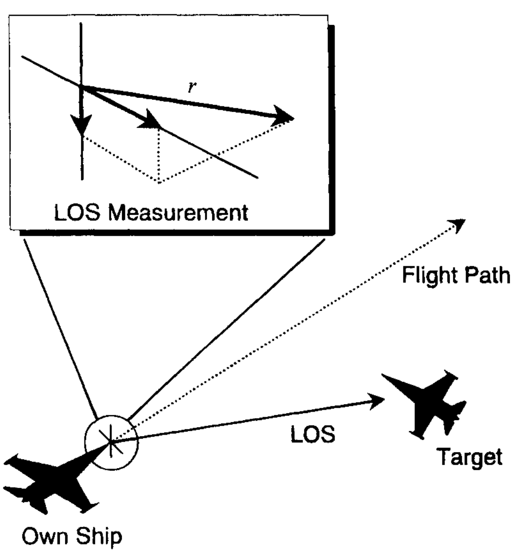

Fig. 1 Bearings-only estimation problem.

Poor initial state knowledge and inherent system nonlinearity makes the design of a good recursive estimator difficult. Design of a quickly coverging recusive filter which is free from lockup and divergence even when it is initialized poorly is the specific problem addressed in this paper.

III. Line-of-Sight Algebraic Relationships

Certain matrix forms and vector-matrix relationships are used in the mathematical development of the sequel and have enlightening geometric significance.

The two LOS projection matrices are: rr^T—the rank 1 idempotent projection matrix, which projects all vectors onto the LOS vector, and $I - rr^T$—the rank 2 idempotent projection matrix, which projects all vectors onto the space orthogonal to the LOS vector.

The following vector relationships are useful:

$$R = |R|r \tag{1}$$

$$|R| = r^T R \tag{2}$$

$$r^T r = 1 \tag{3}$$

$$(rr^T)r = r \tag{4}$$

$$(I - rr^T)r = 0 \tag{5}$$

$$\dot{r} = |R|^{-1}(I - rr^T)\dot{R} \tag{6}$$

$$\dot{r}^T r = 0 \tag{7}$$

$$(rr^T)\dot{r} = 0 \tag{8}$$

$$(I - rr^T)\dot{r} = \dot{r} \tag{9}$$

$$\dot{R} = \dot{r}|R| + r\frac{d|R|}{dt} \tag{10}$$

$$\frac{d|R|}{dt} = r^T \dot{R} \tag{11}$$

Relationships (1–5) are evident from the definitions of r and R. Relationships (6–11) are derived in Appendix A.

IV. Filter Structure

A. Filter Equations in the Modified Polar Coordinate System

The components of the state vector of the conventional MP systems are LOS angle, LOS angle rate, reciprocal range, and range rate divided by range. In the spirit of the algebraic approach presented in this paper the components are defined:

$$x_{MP} \equiv \begin{pmatrix} x_{MP(1)} \\ x_{MP(2)} \\ x_{MP(3)} \\ x_{MP(4)} \end{pmatrix} \equiv \begin{pmatrix} r \\ \dot{r} \\ \dfrac{1}{|R|} \\ \dfrac{1}{|R|}\dfrac{d|R|}{dt} \end{pmatrix} \tag{12}$$

The state dynamics, derived in Appendix B, are given by differentiating each component of state vector x_{MP}

$$\dot{x}_{MP} = f(x_{MP}) + g(x_{MP})\,(u_{os} - u_{tg}) \tag{13}$$

where

$$f(x_{MP}) = \begin{pmatrix} x_{MP(2)} \\ -2x_{MP(4)}x_{MP(2)} - (x^T_{MP(2)}x_{MP(2)})x_{MP(1)} \\ -x_{MP(3)}x_{MP(4)} \\ (x^T_{MP(2)}x_{MP(2)}) - x^2_{MP(4)} \end{pmatrix} \quad (14)$$

$$g(x_{MP}) = -\begin{pmatrix} 0_{3\times 3} \\ x_{MP(3)}(I - x_{MP(1)}x^T_{MP(1)}) \\ 0_{1\times 3} \\ x_{MP(3)}x^T_{MP(1)} \end{pmatrix} \quad (15)$$

By defining the measurement as the direction cosine relative to known axes the measurement equation is linear

$$y = h(x) = Cx + w \quad (16)$$

$$C = (a^T \quad 0_{2\times 3} \quad 0_{2\times 1} \quad 0_{2\times 1}) \quad (17)$$

B. Filter State Propagation

The MP state dynamics, presented in Eq. (13), are nonlinear. Proper extrapolation of the state estimate is normally performed by numerical integration. It is shown[3] that although the MP state dynamics are nonlinear they can be integrated in closed form. This integration is achieved by transforming the MP state to Cartesian coordinates, extrapolating the state linearly, then transforming back to MP coordinates. The transformations of the state vectors between the different coordinate systems is nonlinear, but exact. Consequently there is no loss of information.

The Cartesian state vector is defined

$$x_c \equiv \begin{pmatrix} x_{c(1)} \\ x_{c(2)} \end{pmatrix} \equiv \begin{pmatrix} R \\ \dot{R} \end{pmatrix} \quad (18)$$

The dynamics of the state vector in Cartesian coordinates is linear and is given by

$$\dot{x}_c = \begin{pmatrix} x_{c(2)} \\ 0_{3\times 1} \end{pmatrix} + \begin{pmatrix} 0_{3\times 3} \\ -I_3 \end{pmatrix}(u_{os} - u_{tg}) \quad (19)$$

With the assumption $E[u_{tg}] = 0$ the discrete-time stochastic extrapolation is written

$$\tilde{x}_c[(k+1)T] = \Phi\hat{x}_c(kT) + \Gamma u_{os} \quad (20)$$

where

$$\Phi = \begin{pmatrix} I_3 & I_3T \\ 0_{3\times 3} & I_3 \end{pmatrix}, \qquad \Gamma = -\begin{pmatrix} I_3\left(\dfrac{T^2}{2}\right) \\ I_3T \end{pmatrix} \quad (21)$$

The conversion from the MP system to Cartesian system is given by

$$x_c = f^c_{MP}(x_{MP}) = \begin{pmatrix} |R|r \\ r\left(\dfrac{d|R|}{dt}\right) + \dot{r}|R| \end{pmatrix} \quad (22)$$

$$= \begin{pmatrix} \dfrac{x_{MP(1)}}{x_{MP(3)}} \\ \dfrac{x_{MP(4)}}{x_{MP(3)}}x_{MP(1)} + \dfrac{x_{MP(2)}}{x_{MP(3)}} \end{pmatrix}$$

The conversion from the Cartesian system to the MP system is given by

$$x_{MP} = f^{MP}_c(x_c) = \begin{pmatrix} \dfrac{R}{|R|} \\ \dfrac{1}{|R|}(I - rr^T)\dot{R} \\ \dfrac{1}{|R|} \\ \dfrac{1}{|R|}\dfrac{d|R|}{dt} \end{pmatrix} = \begin{pmatrix} \dfrac{R}{|R|} \\ \dfrac{1}{|R|}\left(I - \dfrac{RR^T}{R^TR}\right)\dot{R} \\ \dfrac{1}{|R|} \\ \dfrac{R^T\dot{R}}{R^TR} \end{pmatrix} \quad (23)$$

$$= \begin{pmatrix} \dfrac{x_{c(1)}}{|x_{c(1)}|} \\ \dfrac{1}{|x_{c(1)}|}\left(I - \dfrac{x_{c(1)}x^T_{c(1)}}{x^T_{c(1)}x_{c(1)}}\right)x_{c(2)} \\ \dfrac{1}{|x_{c(1)}|} \\ \dfrac{x^T_{c(1)}x_{c(2)}}{x^T_{c(1)}x_{c(1)}} \end{pmatrix}$$

Combining Eqs. (20), (22), and (23), the closed-form MP state extrapolation estimate extrapolation proposed in previous work,[3] extended to three dimensions, is stated succinctly

$$\tilde{x}_{MP}[(k+1)T] = f^{MP}_c\{\Phi f^c_{MP}[\hat{x}_{MP}(kT)] + \Gamma u_{os}\} \quad (24)$$

C. State Covariance Propagation

State extrapolation in the Cartesian system is linear so that the state covariance extrapolation is given by the conventional covariance extrapolation equation

$$\tilde{P}_c[(k+1)T] = \Phi\hat{P}_c(kT)\Phi^T + \Gamma V\Gamma^T \quad (25)$$

D. State and State Covariance Update

The state and state covariance update are performed in the MP system. The a priori MP state covariance matrix is calculated from the a priori Cartesian state covariance matrix

$$\tilde{P}_{MP} = J^{MP}_c\tilde{P}_c(J^{MP}_c)^T \quad (26)$$

where $J^{MP}_c = \partial x_{MP}/\partial x_c$,

$$J^{MP}_c = \begin{pmatrix} |R|^{-1}(I - rr^T) & 0_{3\times 3} \\ J_{(2,1)} & |R|^{-1}(I - rr^T) \\ -|R|^{-2}r^T & 0_{3\times 3} \\ J_{(4,1)} & -|R|^{-1}r^T \end{pmatrix}$$

$$J_{(2,1)} = -\frac{1}{|R|}(\dot{r}r^T + r\dot{r}^T) - \frac{1}{|R|^2}\frac{d|R|}{dt}(I - rr^T) \quad (27)$$

$$J_{(4,1)} = \frac{1}{|R|}\left[\dot{r}^T - \left(\frac{1}{|R|}\frac{d|R|}{dt}\right)r^T\right]$$

The measurement Eq. (16) in MP coordinates is linear in state so the update assumes the conventional Kalman filter form

$$K = \tilde{P}_{MP}C^T(C\tilde{P}_{MP}C^T + W)^{-1}$$

$$\hat{x}_{MP} = \tilde{x}_{MP} + K(y - C\tilde{x}_{MP})$$

$$\hat{P}_{MP} = (I - KC)\tilde{P}_{MP}(I - KC)^T + KWK^T$$

After the update of $\tilde{x}_{MP}$ and $\tilde{P}_{MP}$ the a posteriori quantities are converted back to Cartesian coordinates to perform the next extrapolation. The state $\hat{x}_{MP}$ is converted to $\hat{x}_c$ according to Eq. (22). $\hat{P}_{MP}$ is converted to $\hat{P}_c$:

$$\hat{P}_c = J^c_{MP}\hat{P}_{MP}(J^c_{MP})^T \tag{28}$$

where $J^c_{MP} = \partial x_c / \partial x_{MP}$,

$$J^c_{MP} = \begin{pmatrix} |R| I_3 & 0_{3\times 3} & -|R|^2 r & 0_{3\times 1} \\ \left(\frac{d|R|}{dt}\right) I_3 & |R| I_3 & J_{(2,3)} & |R| r \end{pmatrix} \tag{29}$$

where

$$J_{(2,3)} = -|R|^2 \left[\dot{r} + \left(\frac{1}{|R|}\frac{d|R|}{dt}\right) r\right]$$

E. Iterating the Extended Kalman Filter about the Update

Combining the MP and Cartesian coordinate systems allows for the superposition of this nonlinear filter problem onto the structure of a linear Kalman filter. The nonlinear approximations of the problem are implicitly contained in the Jacobians defined by Eq. (27) and (29). These Jacobians are stochastic estimates of the true Jacobians. Their use introduces error into the filter.

Preliminary results have shown that some filter improvement is achieved by iteratively calculating J^{MP}_c about the a posteriori state estimate. This improvement is achieved with one or two iterations (two or three times through the update loop). More iterations appear to be detrimental to filter performance.

F. Filter Initialization

The filter state is initialized from the first measurement of the target LOS unit vector and from available a priori target range and target velocity information. This initialization vector is expressed in Cartesian coordinates

$$x_c(k=0) = \begin{pmatrix} r|R|_0 \\ (\dot{R})_0 \end{pmatrix} \tag{30}$$

The initial state covariance is calculated by forming the outer product of the estimated initial state errors. The cross-correlation, off-diagonal terms are nulled out in accordance with the assumption that the initial estimation errors of R and $\dot{R}$ are uncorrelated.

$$\delta x_c = \begin{pmatrix} r\delta|R|_0 + \delta r|R|_0 \\ \delta v_{tg} \end{pmatrix} = \begin{pmatrix} \delta x_{c(1)} \\ \delta x_{c(2)} \end{pmatrix} \tag{31}$$

and

$$P_c(k=0) = \begin{pmatrix} \delta x_{c(1)}\delta x^T_{c(1)} & 0_{3\times 3} \\ 0_{3\times 3} & \delta x_{c(2)}\delta x^T_{c(2)} \end{pmatrix} \tag{32}$$

V. Simulation and Design Optimization

The Monte Carlo simulations provide the statistical characterization of the filter necessary to optimize design. For Cartesian bearings-only estimation filters, due to the Cauchy-like distribution of the state estimate,[5] it is necessary to tune the initial state covariance to achieve a compromise between the rate of convergence, probability of divergence, and probability of lockup. The hybrid coordinate system filter does not exhibit frequent divergence and lockup and is far less sensitive to initial state covariance assignment than the Cartesian filter is. The statistical performance of the hybrid filter is effected, in a complicated way, by the number of iterations performed in the calculation of the filter Jacobians. The number of iterations as well as some minor initial state covariance tuning is design-specific and performed using statistics provided by the Monte Carlo simulations.

A. Simulation Parameters

The MP-hybrid and Cartesian, bearings-only estimation filters were simulated using a stationary target. The parameters common to all the simulations are listed in Table 1. The initial range error and the number of iterations in the Kalman filter update vary among the simulations. The simulation scenario is illustrated in Fig. 2.

B. The Monte Carlo Method and Confidence Levels

The Kalman filter bearings-only estimation filter does not yield Gaussian-distributed state estimation errors (and Rayleigh-distributed range errors) because the system dynamics are not linear. Occurrances of lockup and divergence, even when infrequent, appear as outliers with a non-Gaussian frequency of occurrance. The ensemble mean and standard deviation are thus insufficient mathematically and inadequate practically for characterizing the statistical performance of the filter.

The complete statistical picture of range error from these Monte Carlo trials is provided by the three-dimensional histogram for which the bin content is a function of both time and range error. The $M\%$ confidence level corresponds to a minimum range error, $\epsilon_{M\%}$, below which are the errors of $M\%$ of all the Monte Carlo trials.

For these simulations the confidence levels were generated by sorting the Monte Carlo range errors from the 1000 trials for each time t_i. The 50% confidence level at time t_i corresponds to the range error of the entry number 500 of the sorted list. Likewise, the 90% and 97% confidence level corresponds to entry 900 and 970 of the sorted list, respectively. The convergence graphs of the 50%, 90%, and 97% confidence level represents the time histories of the range error corresponding to entry 500, 900, and 970, respectively.

Table 1 Monte Carlo simulation parameters

Target dynamics	Fixed target	0 m/s
Own-ship dynamics	Constant velocity	231 m/s
Own-ship altitude	Constant	300 m
Azimuth sensor noise	Normal distribution	1 deg at 0 deg incidence
Azimuth sensor bias		0 deg
Initial azimuth		45 deg
Initial range		46300 m
Initial range estimate		83340 m
Filter update rate		0.5 s
Monte Carlo trials		1000 trials

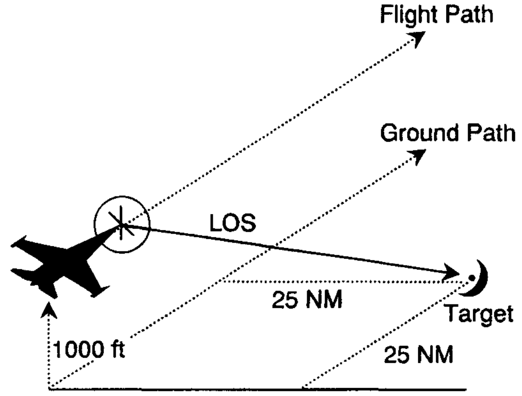

Fig. 2 Simulation scenario.

C. Hybrid Filter Simulation Results

Figure 3 shows the convergence of the 50%, 90%, and 97% confidence levels of the hybrid filter. The filter is implemented with one iteration (twice through the update loop) and the initial range estimate error is 170%. Figure 4 is an enlargement of Fig. 3.

Figure 5 shows the convergence of the 90% confidence level parametrically as a function of the number of iterations in the update loop. Two or three iterations improve the convergence of the 90% confidence level.

Figure 6 shows the convergence of the 97% confidence level as a function of the number of iterations. These results are from the same Monte Carlo runs as that of Fig. 5. The filters with two and three iterations fail to converge and lockup with a gross error. The filters with zero and one iteration converge without bias. The filter with one iteration is thus found empirically to provide the best compromise between filter convergence performance and filter stability.

D. Hybrid Filter Comparative Stability

Figure 7 is a single run showing the Cartesian filter lockup even when the filter is initialized correctly but has a large initial range covariance. In comparison, the hybrid filter conveges without bias.

Figure 8 is a single run showing the divergence of the Cartesian filter for gross errors in the initial range estimate and a large initial range covariance. Single-run simulation parameters are shown in Table 2.

VI. Discussion and Conclusion

The algorithm described in this paper has been simulated on a number of scenarios and has demonstrated good convergence and robustness to large errors in the initial range guess. Filter lockup and divergence phenomena, common to EKFs has not been observed. One iteration in the update loop for the calculation of observation partials is found empirically to improve filter performance. Using more than two iterations is found to be detrimental to filter performance.

Operation counts, not shown here, are only slightly greater than other bearings-only estimation filters. This filter requires no calculation of trigonometric functions which more than compensates for the slight increase in operation counts.

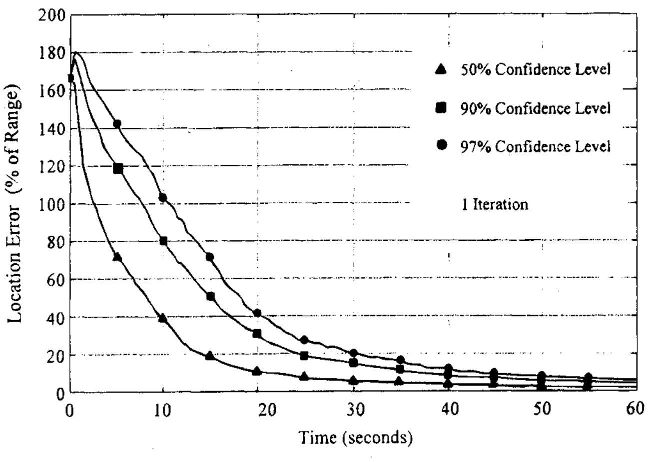

Fig. 3 Convergence of 50%, 90%, and 97% confidence levels for hybrid filter.

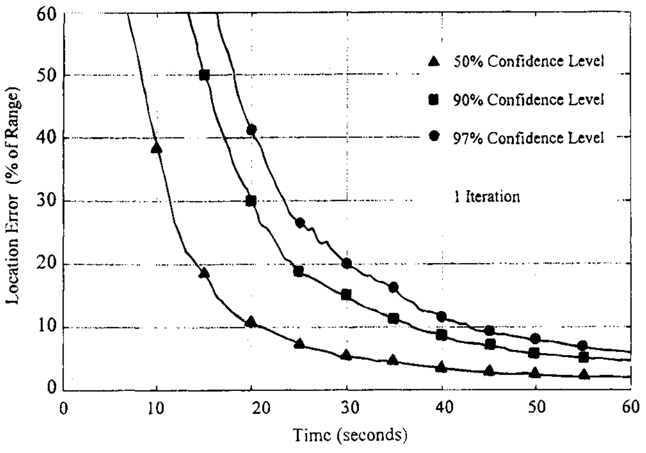

Fig. 4 Magnification of convergence curves of Fig. 3.

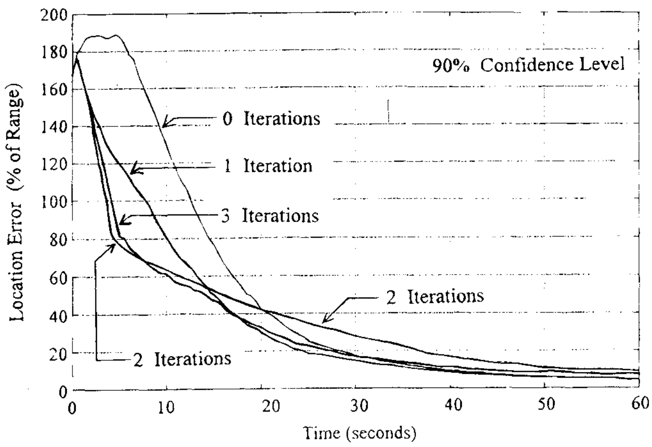

Fig. 5 Convergence of 90% confidence level as a function of update loop iterations.

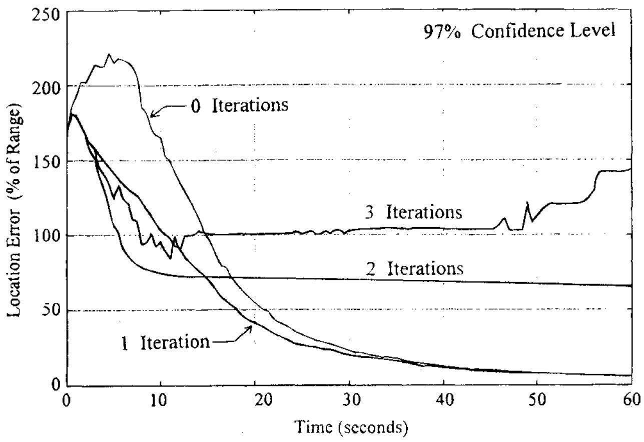

Fig. 6 Convergence of 97% confidence level as a function of update loop iterations.

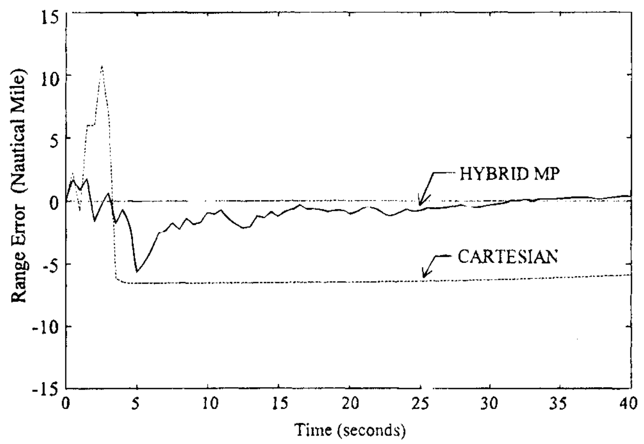

Fig. 7 Hybrid filter converges and Cartesian filter locks up.

Table 2 Simulation parameters for single-run comparison of hybrid and Cartesian filters

	Figure 7	Figure 8
Target dynamics	0 m/s	0 m/s
Own-ship dynamics	231 m/s	231 m/s
Own-ship altitude	300 m	300 m
Azimuth sensor noise	1 deg at 0 deg incidence	1 deg at 0 deg incidence
Azimuth sensor bias	0 deg	0 deg
Initial azimuth	45 deg	45 deg
Initial range	46,300 m	46,300 m
Initial range estimate	185,200 m	83,340 m
Initial range standard deviation	37,040 m	138,900 m
Filter update rate	0.5 s	0.5 s

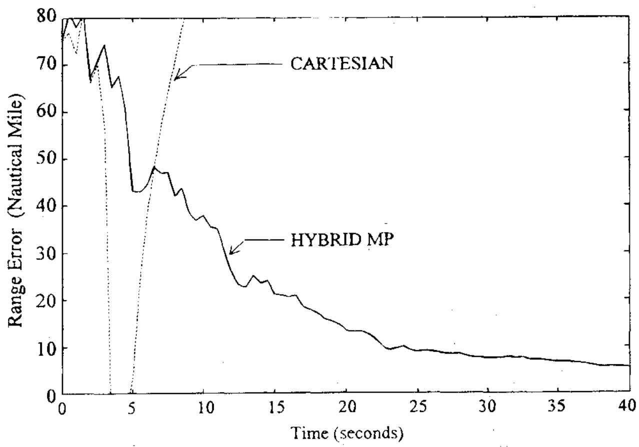

Fig. 8 Hybrid filter converges and Cartesian filter diverges.

Appendix A: Line-of-Sight Algebra

The relationships expressed in Eqs. (1–11) form a basis for a line-of-sight algebra which is very useful for the derivation and manipulation of the tracking filter equations.

This algebra is based upon the LOS coordinate system formed by the unit LOS vector r and the projection of $\dot R$ upon the plane normal to r, i.e., $(I - rr^T)\dot R$. The third direction can be constructed using the cross product $r \times (I - rr^T)\dot R$, though this calculation is usually not necessary.

The two projection matrices rr^T and $I - rr^T$ occur frequently and are particularly significant. The matrix rr^T is a rank 1 idempotent projection matrix which projects all vectors onto the LOS. The matrix $I - rr^T$ is a rank 2 idempotent projection matrix, is the orthogonal complement of rr^T, and projects all vectors onto the planar space orthogonal to the LOS.

The relationships expressed by Eqs. (1–5) are derived from the definitions of r, R, and $|R|$. The relationships expressed by Eqs. (6–11) are very useful and are derived as follows:

Proof of Eq. (11)

$$|R|^2 = R^T R$$

$$2|R|\left(\frac{\mathrm{d}|R|}{\mathrm{d}t}\right) = 2R^T\dot R$$

$$\frac{\mathrm{d}|R|}{\mathrm{d}t} = \left(\frac{R}{|R|}\right)^T \dot R = r^T\dot R$$

Proof of Eq. (6)

$$r \equiv \frac{R}{|R|}$$

$$\dot{\mathrm{r}} = \frac{\dot{\mathrm{R}}}{|\mathrm{R}|} - \frac{\mathrm{R}}{|\mathrm{R}|^2}\left(\frac{\mathrm{d}|R|}{\mathrm{d}t}\right)$$

$$\dot r = |R|^{-1}\dot R - |R|^{-2}R(r^T\dot R)$$

$$\dot r = |R|^{-1}\dot R - |R|^{-1}rr^T\dot R$$

$$\dot r = |R|^{-1}(I - rr^T)\dot R$$

Proof of Eq. (7)

$$\dot r^T r = |R|^{-1}\dot R^T(I - rr^T)r$$

$$(I - rr^T)r = 0$$

$$\Rightarrow \dot r^T r = 0$$

Proof of eq. (8) is given:

$$(rr^T)\dot r = r(r^T\dot r) = 0$$

Proof of Eq. (9) is given:

$$(I - rr^T)\dot r = \dot r - r(r^T\dot r) = \dot r$$

Proof of Eq. (10) is given:

$$\dot R = \frac{\mathrm{d}R}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(r|R|) = \dot r|R| + r\frac{\mathrm{d}|R|}{\mathrm{d}t}$$

Appendix B: Derivation of the Filter State Dynamics

As an example of the utility of the algebraic approach presented in this paper, the MP filter dynamics of Eq. (13) are derived. Other derivations presented in this paper use similar techniques.

The derivative of x_{MP} is calculated by differentiating Eq. (12)

$$\dot x_{\mathrm{MP}} = \begin{pmatrix}\dot x_1\\ \dot x_2\\ \dot x_3\\ \dot x_4\end{pmatrix} = \begin{pmatrix} r\\ \dot r\\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{1}{|R|}\right)\\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{1}{|R|}\dfrac{\mathrm{d}|R|}{\mathrm{d}t}\right)\end{pmatrix} = \begin{pmatrix}x_2\\ f_2\\ f_3\\ f_4\end{pmatrix}$$

By definition $\dot x_1 = x_2$. The functions f_2, f_3, and f_4 are derived as follows.

Derivation of f_2:

$$f_2 = \ddot r = \frac{\mathrm{d}\dot r}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{|R|}(I - rr^T)\dot R\right]$$

$$= -\frac{1}{|R|^2}(I - rr^T)\dot R\left(\frac{\mathrm{d}|R|}{\mathrm{d}t}\right) - \frac{1}{|R|}\frac{\mathrm{d}(rr^T)}{\mathrm{d}t}\dot R$$

$$+ |R|^{-1}(I - rr^T)\ddot R$$

$$= -\frac{1}{|R|^2}(I - rr^T)\dot R\left(\frac{\mathrm{d}|R|}{\mathrm{d}t}\right)$$

$$-|R|^{-1}(r\dot r^T + \dot r r^T)\dot R + |R|^{-1}(I - rr^T)\ddot R$$

Noting $\dot R = r\dfrac{\mathrm{d}|R|}{\mathrm{d}t} + \dot r|R|$

$$\ddot r = -\frac{1}{|R|^2}(I - rr^T)\left(r\frac{\mathrm{d}|R|}{\mathrm{d}t} + \dot r|R|\right)\left(\frac{\mathrm{d}|R|}{\mathrm{d}t}\right)$$

$$-\frac{1}{|R|}(r\dot r^T + \dot r r^T)\left(r\frac{\mathrm{d}|R|}{\mathrm{d}t} + \dot r|R|\right) + |R|^{-1}(I - rr^T)\ddot R$$

Noting further $(I - rr^T)\dot r = \dot r$, $(I - rr^T)r = 0$, $\dot r^T r = 0$, and $r^T r = 1$

$$\ddot{r} = -\dot{r}\left(\frac{1}{|R|}\frac{d|R|}{dt}\right) - \frac{1}{|R|}\left[r(\dot{r}^T\dot{r})|R| + \dot{r}\left(\frac{d|R|}{dt}\right)\right]$$
$$+|R|^{-1}(I - rr^T)\ddot{R}$$
$$= -\dot{r}\left(\frac{2}{|R|}\frac{d|R|}{dt}\right) - r(\dot{r}^T\dot{r}) + \frac{1}{|R|}(I - rr^T)\ddot{R}$$

The LOS acceleration vector is the difference between the target and own-ship accelerations, i.e.,

$$\ddot{R} = u_{tg} - u_{os}$$

Therefore,

$$\ddot{r} = -\dot{r}\left(\frac{2}{|R|}\frac{d|R|}{dt}\right) - r(\dot{r}^T\dot{r}) + \frac{1}{|R|}(I - rr^T)(u_{tg} - u_{os})$$

Substituting the MP state variable definitions from Eq. (12) yields

$$f_2 = \dot{x}_2 = \ddot{r} = -2x_2x_4 - x_1(x_2^Tx_2)$$
$$+x_3(I - x_1x_1^T)(u_{tg} - u_{os})$$

Derivation of f_3:

$$f_3 = \dot{x}_3 = \frac{d}{dt}\left(\frac{1}{|R|}\right)$$
$$= -\frac{1}{|R|^2}\frac{d|R|}{dt}$$
$$= = -\frac{1}{|R|}\left(\frac{1}{|R|}\frac{d|R|}{dt}\right)$$
$$= -x_3x_4$$

Derivation of f_4:

$$f_4 = \dot{x}_3 = \frac{d}{dt}\left(\frac{1}{|R|}\frac{d|R|}{dt}\right)$$
$$= -\frac{1}{|R|^2}\left(\frac{d|R|}{dt}\right)^2 + \frac{1}{|R|}\frac{d}{dt}\left(\frac{d|R|}{dt}\right)$$

The range rate is given by Eq. (11)

$$\frac{d|R|}{dt} = r^T\dot{R}$$
$$\Rightarrow \dot{x}_4 = -\frac{1}{|R|^2}\left(\frac{d|R|}{dt}\right)^2 + \frac{1}{|R|}\frac{d}{dt}(r^T\dot{R})$$
$$= -\frac{1}{|R|^2}\left(\frac{d|R|}{dt}\right)^2 + \frac{1}{|R|}(\dot{r}^T\dot{R} + r^T\ddot{R})$$

Noting $\dot{R} = r\dfrac{d|R|}{dt} + \dot{r}|R|$ and $\dot{r}^Tr = 0$,

$$\dot{x}_4 = -\frac{1}{|R|^2}\left(\frac{d|R|}{dt}\right)^2$$
$$+\frac{1}{|R|}\left[\dot{r}^T\left(r\frac{d|R|}{dt} + \dot{r}|R|\right) + r^T\ddot{R}\right]$$
$$= -\frac{1}{|R|^2}\left(\frac{d|R|}{dt}\right)^2 + \frac{1}{|R|}[(\dot{r}^T\dot{r})]|R| + r^T\ddot{R}]$$

Again, $\ddot{R} = u_{tg} - u_{os}$ and f_4 is given by

$$f_4 = \dot{x}_4 = -x_4^2 + x_2^Tx_2 + x_3x_1^T(u_{tg} - u_{os})$$

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